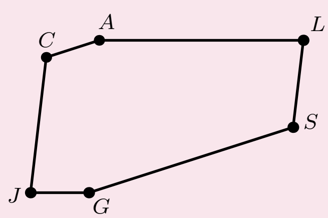
PoTW 1: Week of 5-27-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #1: Vexing Hexagon

Topic: Geometry

Hexagon $ALSGJC$ has the curious property that all of its opposite sides are parallel; that is, $\overline{AL} \parallel \overline{GJ}$, $\overline{LS} \parallel \overline{JC}$, and $\overline{SG} \parallel \overline{CA}$. Suppose that $\overline{AL} = 21\sqrt{2}$, $\overline{LS} = 9\sqrt{3}$, $\overline{SG} = 18\sqrt{3}$, $\overline{GJ} = 9\sqrt{2}$, $\overline{JC} = 14\sqrt{3}$, and $\overline{CA} = 7\sqrt{3}$. If R is the length of the circumradius of $\triangle ASJ$, compute $(R^2 - 378)$.



Source: CMC Mock ARML, by Eric Shen.

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Before diving into the solution, we begin with a brief segue on Power of a Point, which not only ends up being the crux of our problem, but more generally is extremely useful for a wide variety of different geometric configurations. The way that the theorem is normally formulated looks something like the following:

> **Power of a Point:** Consider a circle ω, and a point Q which does not lie on ω. Let AB and CD be two chords of ω such that Q lies on the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Then
>
> $$QA \cdot QB = QC \cdot QD.$$

Note that this definition encompasses all possible cases: if Q lies outside of ω, then we have two secants (or tangents, in the cases that point A is equal to point B, or point C is equal to point D); and, if Q lies inside ω, then we have two intersecting chords.

Of particular interest to us is the invariance of the quantity $QA \cdot QB$ encoded in the statement of the theorem; in other words, regardless of the location of the chord AB with respect to ω, the theorem tells us that this quantity will always be constant. Therefore, for any point Q, we can call the value of this quantity as the *power* of Q with respect to ω, which we can denote as $P(Q, \omega)$.

Because this quantity is the same for any arbitrary chord that we choose in ω, we can calculate it by choosing the chord which lies on the same line determined by Q and the center of ω, call O. Let R denote the radius of ω. Then, using this chord, we get that

$$P(Q, \omega) = (OQ + R)(OQ - R) = OQ^2 - R^2,$$

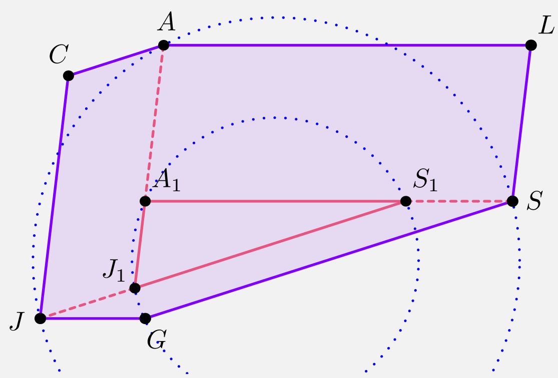
where $P(Q, \omega)$ is negative if R lies inside ω, zero if R lies on ω, and positive if R lies outside of ω.

Reformulating the original PoP theorem in terms of $P(Q, \omega)$ vastly expands the usage of the theorem and its applicability to more complex problems. For our purposes, we only examine one particular corollary of this reformulation:

> **Corollary:** Any set of points $\mathcal{H}$ which all have the same power with respect to ω lie on a circle concentric to ω.

The proof follows naturally from the definition of $P(Q, \omega)$. For any point $A \in \mathcal{H}$, because we must have $P(A, \omega) = OA^2 - R^2$, for the power to be constant, we simply require that OA be constant. In other words, we must have that all the points in $\mathcal{H}$ lie at a fixed radius from O, which concludes our proof. We are now ready to present our solution.

Solution (intended, by the problem author):
This solution is also equivalent to the solution submitted by ***Jessica He****!*



Construct points A_1, S_1, and J_1 in the interior of our hexagon, so that $ALSA_1$, $SGJS_1$, and $JCAJ_1$ are parallelograms. Let ω be the circumcircle of $\triangle A_1S_1J_1$, and observe that we have

$$P(A,\omega) = AA_1 \cdot AJ_1 = LS \cdot JC = 378,$$
$$P(J,\omega) = JJ_1 \cdot JS_1 = CA \cdot SG = 378,$$
$$P(S,\omega) = SS_1 \cdot SA_1 = GJ \cdot AL = 378.$$

It follows that the circumcircle of $\triangle ASJ$ is concentric to ω (by our corollary above). Therefore, if we let r denote the length of the radius of ω, then we have

$$378 = P(A,\omega) = R^2 - r^2 \implies R^2 - 378 = r^2,$$

so it suffices to compute r^2. But note that because of the nature of the opposite parallel sides of our hexagon, we have $A_1J_1 = JC - LS = 5\sqrt{3}$, $A_1S_1 = AL - GJ = 12\sqrt{2}$, and $S_1J_1 = SG - CA = 11\sqrt{3}$. Therefore, $\triangle A_1S_1J_1$ is a right triangle with hypotenuse $11\sqrt{3}$, and thus

$$R^2 - 378 = r^2 = \left(\frac{11}{2}\sqrt{3}\right)^2 = \frac{363}{4}.$$