PoTW 2: Week of 6-3-2021

Problem of the Week at shsmathteam.com

Submission form (because this week's problem is lengthy, partial solutions are okay to submit): link to submit

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Problem of the Week #2: Catalan Craziness

Topic: Combinatorics

Suppose that we have a set of $2n$ letters, n of which are X's, and n of which are Y's. Consider any sequence S of all $2n$ letters which obeys the property that, in any of the sub-sequences of S beginning with the first letter, the number of Y's never exceeds the number of X's. For example, for $n = 3$, the following sequences are all valid:

$$XXXYYY$$
$$XXYXXY$$
$$XXYYXY$$
$$XYXXYY$$
$$XYXYXY,$$

because in each sequence, there does not exist a k for which the first k letters of the sequence has more Y's than X's.

It is well known that the number of ways to create such a sequence can be described by the *Catalan numbers*, which obey the following expression:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

For instance, $C_3 = 5$ as depicted in the above example. Although there are many ways to prove this formula, in this problem we will try to motivate a solution which generalizes easily to "higher-order" Catalan numbers (to be defined later).

Suppose that we call any sequence of n X's and n Y's which does not obey the property described above as a *bad* sequence.

(a). Based on the formula for the nth Catalan number, calculate the number of bad sequences of length $2n$, in terms of n.

If done correctly, this should equal the number of sequences of $n-1$ X's, and $n+1$ Y's (all of which are bad). Therefore, in order to prove our formula for the n-th Catalan number, it remains to prove that the number of bad sequences which comprises of n X's and n Y's, is equal to the number of sequences which comprises of $n-1$ X's and $n+1$ Y's.

(b). Any sequence which starts with 1 X and 2 Y's, in some order, must be bad. Calculate the number of sequences comprising of n X's and n Y's which start with 1 X and 2 Y's (in some order), and then do the same for sequences comprising of $n-1$ X's and $n+1$ Y's. Show that these quantities are equal.

(c). Finish the proof.

(helpful hint: use the previous question to motivate the rest of your solution. more specifically, any bad sequence must start with k X's and k+1 Y's in some order).

Having established a formula for the nth Catalan number, now suppose that we extend the definition of what defines a good and bad sequence. Suppose that now we define a good sequence as one with the property that the number of Y's never exceeds *twice* the number of X's in any of the initial subsequences of our sequence, and call these sequences as *2-good* sequences. For example, the number of 2-good sequences which comprise of 2 X's and 4 Y's is 3:

$$XXYYYY$$
$$XYXYYY$$
$$XYYXYY.$$

And now, call the *2nd-order* Catalan number $C_n^{(2)}$ as the number of 2-good sequences which comprise of n X's and $2n$ Y's. For instance, in the above example, we have that $C_2^{(2)} = 3$.

(d). Prove that

$$C_n^{(2)} = \frac{1}{2n+1}\binom{3n}{n}.$$

(helpful hint: the way that we proved the formula for C_n in parts (a), (b), and (c) can be used in the exact same manner to prove this formula.)

(e). Find and prove a formula for $C_n^{(m)}$, the *mth-order* Catalan number, defined as the number of m-good sequences which comprise of n X's and $m \cdot n$ Y's.

(helpful hint: as before, the exact same method in the first four parts of this problem works for this part as well.)