PoTW 4: Week of 6-10-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #4: Phun Physics Phridays

Topic: Geometry/Physics
Source: AoPS

Let square $RUDA$ have center P, and XY be a segment passing through P such that X lies on RU and Y lies on DA. Identify the shape of the locus of all possible centroids of trapezoid $UDYX$.

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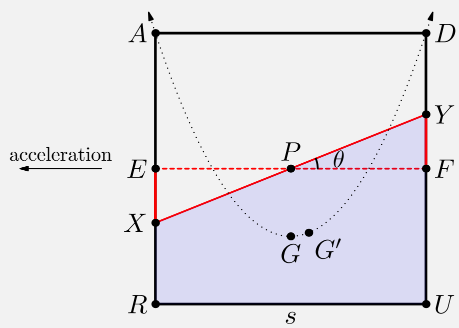
Solution 1 (intended)**:**

We claim that the locus is parabolic.

Suppose that we have a cube filled halfway with water with its bottom face on a flat plane, and that it is accelerating on the plane in a direction perpendicular to one of its faces (call A). Then, the problem is equivalent to proving that the projection of the center of mass of the water onto some face adjacent to A (which isn't the top or bottom face) is a parabola.



This can be proven easily either by directly using the formula for center of mass, or using potential energy. Here we present the latter solution.

Let G be the center of mass of our system at rest, and G' be the center of mass when our system is accelerating, as depicted in the diagram above. Let s be the side length of the cube, ρ be the density of water, and θ be the tilt of the water surface compared to horizontal (as a result of the cube's acceleration). Note that the total mass of our water is $M = \rho s^3/2$.

In the diagram, note that the change in potential energy of the system (from rest) can be calculated by visualizing it as a shift from $\triangle EPX$ to $\triangle FPY$. The mass of the water contained in $\triangle FPY$ is $m = \frac{\rho s^3 \tan\theta}{8}$, and because the center of mass of the triangle is located $1/3$ of the way up from EF, then our total change in potential energy is equal to

$$\Delta U_y = mg\Delta h_y = g\left(\frac{\rho s^3 \tan\theta}{8}\right)\left(2\cdot\frac{s\tan\theta}{6}\right) = \frac{\rho g s^4 \tan^2\theta}{24}.$$

But this is also equal to $Mg(G'_y - G_y)$, and setting these two quantities equal and solving gives us that

$$G'_y - G_y = \frac{s}{12}\tan^2\theta.$$

We can perform the same calculation in the x direction (eg some imaginary gravitational potential is acting right to left), to get that

$$\Delta U_x = mg\Delta h_x = g\left(\frac{\rho s^3 \tan\theta}{8}\right)\left(2\cdot\frac{s}{3}\right) = \frac{\rho g s^4 \tan\theta}{12},$$

and

$$G'_x - G_x = \frac{s}{6}\tan\theta.$$

But now it is clear that the change in the center of mass is parabolic (parametric in $\tan\theta$), so we are done.

For completeness, we will also present a solution framed directly in the perspective of the original problem (i.e, uses math only). It requires knowledge of projective geometry.

Solution 2 (projective geometry, by Albert Wang)**:**

As in the first solution, we aim to prove that the locus is parabolic.

Let G_1 and G_2 be the centroids of $\triangle XRU$ and $\triangle YXU$, respectively. By linearity, $G' \in G_1G_2$. Also note that G_2 is fixed, because UP does not change with respect to varying θ.

Let $P_\infty = RE \cap GP \cap UF$, M_1 be the midpoint of RU, and M_2 be the midpoint of UY.

It is easy to verify that $M_1 = DG_2 \cap GP_\infty$ (similar triangles), and that $M_2 = G'G \cap P_\infty D$ (GG' is the midline of trapezoid $RUYX$). Also, $G_2G' \cap P_\infty P_\infty$ is the point at infinity on the line $G_2G' = G_2G_1$, which lies on M_1M_2 by homothety at X.

Therefore, M_1, M_2, and $G_2G' \cap P_\infty P_\infty$ are collinear, implying by pascal's theorem that $DG_2G'GP_\infty P_\infty$ all lie on a single conic section. Moreover, because P_∞ lies on the parabola through G,A,D (by the focal point/directrix definition of a parabola), it follows that G' must also lie on the same parabola, and we are finished.