PoTW 6: Week of 7-1-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #6: Plushee Paradise

Topic: Combinatorics
Source: classical

Hannah is buying gifts for a special party celebration! Miraculously, despite having 100 friends, she knows that every single one of them loves stuffed animal plushees. She has five varieties of plushees to choose from: *Shirokuma*, *Penguin*, *Tonkatsu*, *Neko*, and *Tokage*. Because she is crunched for time, she randomly selects one of the five plushees to buy for each of her friends. Calculate the expected value of the product of the quantities of each of the five varieties of plushees that she bought (e.g, if she bought 20 of each type, then this product would be equal to 20^5).

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This problem employs an idea which has widespread use in problems pertaining to expected value: the so-called *linearity of expectation*. The linearity of expectation is usually most easily understood in the context of an example problem, so we'll present two different solutions: our first utilizing direct counting methods, and the second utilizing the linearity of expectation.

Solution 1 (no lin-ev)**:**

Note that our problem is equivalent to calculating the expected value of the product $abcde$ given that $a+b+c+d+e=100$. Let X denote our desired quantity for arbitrarily chosen a, b, c, d, and e.

Then, by the definition of expected value, we have that:

$$\begin{aligned}\mathbb{E}(X) &= \sum n \cdot P(X=n) \\ &= \sum_{\substack{a_1+a_2+a_3 \\ +a_4+a_5=100}} a_1a_2a_3a_4a_4a_5 \cdot \left(\frac{100!}{a_1!a_2!a_3!a_4!a_5!}\right)/(5^{100}) \\ &= \left(\frac{100!}{95!\cdot 5^{100}}\right) \sum_{\substack{a_1+a_2+a_3 \\ +a_4+a_5=100}} \frac{95!}{(a_1-1)!(a_2-1)!(a_3-1)!(a_4-1)!(a_5-1)!} \\ &= \left(\frac{100!}{95!\cdot 5^{100}}\right)(1+1+1+1+1)^{95} = \boxed{\frac{100!}{95!\cdot 5^5}}.\end{aligned}$$

Solution 2 (lin-ev)**:**

Consider the sequence $x_1, \ldots, x_{100}$, where x_i represents the type of plushee that Hannah bought for her i-th friend.

Now consider the set of tuples

$$S = \{(x_a, x_b, x_c, x_d, x_e) : 1 \leq a, b, c, d, e \leq 100\},$$

where x_a, x_b, x_c, x_d, and x_e correspond to Hannah picking Shirokuma, Pengium, Tonkatsu, Neko, and Tokage, respectively. Clearly, the size of S is equivalent to our desired answer. Then, by the linearity of expectation, it suffices to calculate the total number of possible tuples, multiplied by the probability that they are in S, which is equal to

$$\frac{100!}{95!} \cdot \left(\frac{1}{5}\right)^5 = \boxed{\frac{100!}{95!\cdot 5^5}}.$$