PoTW 11: Week of 8-5-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #11: OhMo

Topic: Number Theory
Source: 2019 OMO

Let $\{F_n\}_{n\geq 0}$ be the usually defined Fibonacci sequence ($F_0 = 0$, $F_1 = 1$). Suppose that d is the largest positive integer such that, for all integers $n \geq 0$, d divides $F_{n+420} - F_n$. Compute the remainder when d is divided by 130.

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Solution (pad, AoPS)**:**

The answer is 0.

First note that if d divides $F_{420} - F_0$, and $F_{1+420} - F_1$, then d also divides $F_{n+420} - F_n$ for any n, by adding. Now, it suffices to calculate:

$$\begin{aligned}
\gcd(F_{421} - 1, F_{420}) &= \gcd(F_{419} - 1, F_{420}) \\
&= \gcd(F_{419} - 1, F_{418} + 1) \\
&= \gcd(F_{417} - F_3, F_{418} + F_2) \\
&= \gcd(F_{417} - F_3, F_{416} + F_4) \\
&\ \vdots \\
&= \gcd(F_{211} - F_{209}, F_{210} + F_{210}) \\
&= F_{210}.
\end{aligned}$$

To finish, there are many ways to calculate $F_{210} \pmod{130}$. The cleanest is probably to utilize the fact that $F_m | F_n$ for any $m|n$. This gives us that $F_3, F_5, F_7 | F_{210}$, so in fact $2 \cdot 5 \cdot 13 = 130 | F_{210}$ and our answer is 0, as desired.