PoTW 12: Week of 8-12-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #12: Greatest Divisor

Topic: Number Theory
Source: AoPS

Let $f(x)$ be the function returning the greatest proper divisor of any positive integer x; for example, $f(12) = 6$. Find all n such that $n + f(n)$ is a power of 10.

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Solution (AoPS)**:**
This solution is also equivalent to the solution submitted by ***Faizaan Siddique****!*

We claim that the only solution is $n = 75$.

We want to solve the equation

$$n + f(n) = 10^m, \tag{1}$$

for $m \geq 0$. First note that $f(n)$ divides both n and $f(n)$; therefore, $f(n)$ divides 10^m, so $f(n) = 2^a \cdot 5^b$. Let p be the smallest prime divisor of n, so that $f(n) = n/p$.

If $a > 0$, then we must have $p = 2$. Plugging this into Eq 1, we get that $3f(n) = 10^m$, which is a contradiction since any power of 10 is not a multiple of 3. Therefore, $a = 0$, and consequently $f(n) = 5^b$, whence $p \in \{2, 3, 5\}$. One more application of Eq 1 then gives us $5^b(1 + p) = 10^m$, in which we see that $p = 3$ is the only option that works (the others run into divisibility issues). Therefore, $5^b = 5^m \cdot 2^{m-2}$, giving us $m = b = 2$ and thus $n = 3 \cdot 5^2 = 75$ is our only solution.