PoTW 8: Week of 7-15-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #8: Oh Titu...

Topic: Algebra/Geo
Source: Titu

Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

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For this problem, the key is to observe the miraculous identity:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc).$$

The rest of the problem boils down to rote algebra.

Solution:

Rewrite the given equation as

$$x^3 + y^3 + (-1)^3 - 3(x)(y)(-1) = 0.$$

Then, we can factor the left hand side so that

$$(x + y - 1)(x^2 + y^2 + 1 - xy + x + y) = 0.$$

This implies that $x + y = 1$, or $x^2 + y^2 + 1 - xy + x + y = 0$. To deal with the latter equation, we can proceed in one of two ways:

Method 1: Treat the equation as a quadratic in x:

$$x^2 + (1 - y)x + y^2 + y + 1 = 0.$$

Taking the discriminant for real solutions, we see that we must have

$$(y - 1)^2 - 4(y^2 + y + 1) \geq 0$$
$$\iff (y + 1)^2 \leq 0,$$

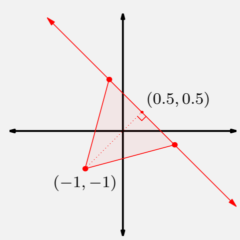
so $x = y = -1$ is our only solution.

Method 2: Multiplying both sides of the equation by 2 and rearranging squared terms, we see that it factors as

$$(x - y)^2 + (x + 1)^2 + (y + 1)^2 = 0,$$

so again our only valid solution is $x = y = -1$.

With either method, we see that the solution set is the line $x + y = 1$ and the point $(-1, -1)$.



The distance between the point and line is equal to $3\sqrt{2}/2$, which is also equal to the height of our equilateral triangle, so our desired area is

$$\frac{\sqrt{3}}{4}\left(\sqrt{6}\right)^2 = \boxed{\frac{3}{2}\sqrt{3}}.$$