PoTW 9: Week of 7-22-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #9: It's Coming Rome

Topic: Combinatorics
Source: ICO 2020

In most European Soccer Leagues, teams follow a double round-robin format, meaning that each team plays every other team in the league exactly twice. In each game, teams receive three points for a win, one point for a draw, and zero points for a loss. At the end of the season, teams are ranked from top to bottom based on their total number of points; if there is a tie, the tiebreaker goes to the teams with the higher *goal differential*, which is equal to the difference between the total number of goals scored and the total number of goals allowed.

Suppose there is a soccer league which has 2021 teams, plays a single round-robin format, and follows the same scoring format described above. Given that there were no draws in the previous season, is it possible for the goal differential of teams ranked top to bottom to have been strictly increasing?

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Solution (intended)**:**

The answer is no. Note that we must have that each team finished the season with a different record; otherwise, tied teams would have been ranked by goal differential contradictory to that stipulated in the problem. Therefore, the first place team must have finished with all wins and a strictly positive goal differential, while the last place team must have finished with all losses and a strictly negative goal differential, so the goal differential from top to bottom can't have been strictly increasing.