PoTW 14: Week of 8-26-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #14: Factor Fest

Topic: Number Theory
Source: Tristan Shin

Solve the equation

$$(a-b)(a^2-b^2)(a^3-b^3)=3c^3$$

in integers a, b, c.

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The key here is to utilize Fermat's Last Theorem.

Solution:

First, expand and simplify the left hand side of the equation:

$$\begin{aligned}(3c)^3 &= 9(a-b)^3(a+b)(a^2+ab+b^2)\\ &= (a-b)^3(9a^3+18a^2b+18ab^2+9b^3)\\ &= (a-b)^3((2a+b)^3+(2b+a)^3)\\ &= [(a-b)(2a+b)]^3+[(a-b)(2b+a)]^3.\end{aligned}$$

Note that this equation takes the form $z^3 = x^3 + y^3$, which has no non-trivial integer solutions in x, y, z by Fermat's last theorem. Therefore, we only need to compute solutions for each of the trivial cases:

Case 1: $(x, y, z) = (0, k, k), (k, 0, k)$

If $a = b$, then we get solutions $(\alpha, \alpha, 0)$, for any integer α. If $(a, b) = (\alpha, -2\alpha)$, then we get that

$$(3c)^3 = (3\alpha)^3(-3\alpha)^3,$$

implying $c = -3\alpha^2$, giving us $(\alpha, -2\alpha, -3\alpha^2)$ as another set of solutions. Similarly, for $(a, b) = (-2\alpha, \alpha)$, we get solutions $(-2\alpha, \alpha, 3\alpha^2)$.

Case 2: $(x, y, z) = (k, -k, 0)$

For this case we require $(a-b)(2a+b)+(a-b)(2b+a) = 0$, which implies that either $a - b = 0$, which is covered in Case 1, or $a = -b$, which gives us $(\alpha, -\alpha, 0)$.

Therefore, our full solution set is the union of $(\alpha, -2\alpha, -3\alpha^2)$, $(-2\alpha, \alpha, 3\alpha^2)$, $(\alpha, \alpha, 0)$, $(\alpha, -\alpha, 0)$ for all integer α.