PoTW 15: Week of 9-2-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #15: two alg practice problems

Topic: Number Theory
Source: David Altizio

(1). Given that $\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{5^k} = \sqrt{5}$, compute the value of the sum

$$\sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{1}{5^k}.$$

(2). Suppose x and y are real numbers such that

$$x^2 + xy + y^2 = 2 \qquad \text{and} \qquad x^2 - y^2 = \sqrt{5}.$$

Calculate all possible distinct values of x.

*For inquiries: andliu22@students.d125.org

Part (1)

Solution 1:

Let S be the value of our desired quantity. Multiplying by 2, we get that

$$\begin{aligned} 2S &= \sum_{k=0}^{\infty} 2\binom{2k+1}{k}\frac{1}{5^k} \\ &= \sum_{k=0}^{\infty} \frac{2k+2}{k+1}\cdot\frac{(2k+1)!}{k!(k+1)!}\frac{1}{5^k} \\ &= 5\sum_{k=0}^{\infty}\binom{2k+2}{k+1}\frac{1}{5^{k+1}} \\ &= 5\left(\sum_{i=0}^{\infty}\binom{2i}{i}\frac{1}{5^i}-1\right), \end{aligned}$$

so $S = \frac{5(\sqrt{5}-1)}{2}$.

Solution 2:
Solution submitted by ***Jeffrey Chen****!*

As in our first solution, let S be the value of our desired quantity. We have that

$$\begin{aligned} S &= \sum_{k=0}^{\infty}\binom{2k+1}{k}\frac{1}{5^k} \\ &= 2\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{5^k} - \frac{1}{k+1}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{5^k}. \end{aligned}$$

Recognize that the right-hand sum mimics the generating function for the Catalan Numbers, which is given by

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{n+1}x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Plugging in $x = \frac{1}{5}$, we get

$$S = 2\sqrt{5} - \frac{1-\sqrt{1/5}}{2/5} = \frac{5\sqrt{5}-5}{2},$$

as desired.

Part (2)

Sketch (Benq)**:**

Let $a = x + y$, $b = x - y$, whence the two given equations can be rewritten $3a^2 + b^2 = 8$, $ab = \sqrt{5}$. Solving gives $(a, b) = (\pm\sqrt{\frac{5}{3}}, \pm\sqrt{3}), (\pm 1, \pm\sqrt{5})$. These can be used to solve for x, some of which may be extraneous.