PoTW 16: Week of 9-9-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #16: 777

Combinatorics
Source: 2004 AIME II/14

Consider a string of n 7's, $7777\cdots77$, into which $+$ signs are inserted to produce an arithmetic expression. For example, $7+77+777+7+7=875$ could be obtained from eight 7's in this way. For how many values of n is it possible to insert $+$ signs so that the resulting expression has value 7000?

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Solution (intended)**:**
Solution equivalent to the submissions by ***Faizaan Siddique****,* ***Shaurya Agrawal****, and* ***Jaden Chen****!*

Divide through by 7, so that we seek all n such that

$$\begin{aligned} 111a + 11b + c &= 1000 \\ 3a + 2b + c &= n. \end{aligned}$$

Let S_k denote the set of all possible values of n when $a = k$, for $0 \leq k \leq 9$. Now, our central claim:

Claim.

$$S = \bigcup_{k=0}^{8} S_k = \{1000 - 9k | 0 \leq k \leq 106\}.$$

Proof (main idea). First note that n is invariant modulo 9, owing to the fact that $111a \equiv 3a$ and $11b \equiv 2b$ modulo 9. It follows that $n \equiv 1000 \pmod 9$. Note also that the maximum element in S is when $c = 1000$, giving $n = 1000$; similarly, the minimum element in S is when $(a, b, c) = (8, 10, 2)$, giving $n = 46$. It thus remains to verify that the unions fully cover our desired residue class, which is not too hard with casework. $\square$

We have 107 values of n in S. Our final case is when $a = 9$, which gives us the additional solution $n = 28$, for a total of $107 + 1 = \boxed{108}$ possible values of n.