Fall 2021 AMC 12B SOLUTIONS

Stevenson Math Team*

November 2021

Contents

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0 Problems

ANSWER KEY: EBCEC CDDBE CBEBE BAAEA ACACC

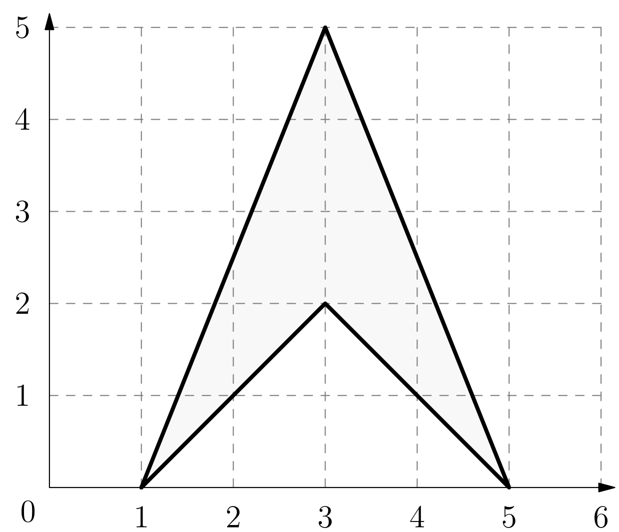
Problem 1.

What is the value of $1234 + 2341 + 3412 + 4123$?

(A) $10,000$ **(B)** $10,010$ **(C)** $10,110$ **(D)** $11,000$ **(E)** $11,110$

Problem 2.

What is the area of the shaded figure shown below?



(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

Problem 3.

At noon on a certain day, Minneapolis is N degrees warmer than St. Louis. At 4:00 the temperature in Minneapolis has fallen by 5 degrees while the temperature in St. Louis has risen by 3 degrees, at which time the temperatures in the two cities differ by 2 degrees. What is the product of all possible values of N?

(**A**) 10 (**B**) 30 (**C**) 60 (**D**) 100 (**E**) 120

Problem 4.

Let $n = 8^{2022}$. Which of the following is equal to $\frac{n}{4}$?

(A) 4^{1010} **(B)** 2^{2022} **(C)** 8^{2018} **(D)** 4^{3031} **(E)** 4^{3032}

Problem 5.

Call a fraction $\frac{a}{b}$, not necessarily in the simplest form special if a and b are positive integers whose sum is 15. How many distinct integers can be written as the sum of two, not necessarily different, special fractions?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

Problem 6.

The largest prime factor of 16384 is 2, because $16384 = 2^{14}$. What is the sum of the digits of the largest prime factor of 16383?

(A) 3 **(B)** 7 **(C)** 10 **(D)** 16 **(E)** 22

Problem 7.

Which of the following conditions is sufficient to guarantee that integers x, y, and z satisfy the equation

$$x(x-y)+y(y-z)+z(z-x)=1?$$

(A) $x > y$ and $y = z$ **(B)** $x = y - 1$ and $y = z - 1$ **(C)** $x = z + 1$ and $y = x + 1$
(D) $x = z$ and $y - 1 = x$ **(E)** $x + y + z = 1$

Problem 8.

The product of the lengths of the two congruent sides of an obtuse isosceles triangle is equal to the product of the base and twice the triangle's height to the base. What is the measure, in degrees, of the vertex angle of this triangle?

(A) 105 **(B)** 120 **(C)** 135 **(D)** 150 **(E)** 165

Problem 9.

Triangle ABC is equilateral with side length 6. Suppose that O is the center of the inscribed circle of this triangle. What is the area of the circle passing through A, O, and C?

(A) 9π **(B)** 12π **(C)** 18π **(D)** 24π **(E)** 27π

Problem 10.

What is the sum of all possible values of t between 0 and 360 such that the triangle in the coordinate plane whose vertices are $(\cos 40^\circ, \sin 40^\circ)$, $(\cos 60^\circ, \sin 60^\circ)$, and $(\cos t^\circ, \sin t^\circ)$ is isosceles?

(A) 100 **(B)** 150 **(C)** 330 **(D)** 360 **(E)** 380

Problem 11.

Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

(A) $\frac{3}{4}$ **(B)** $\frac{57}{64}$ **(C)** $\frac{59}{64}$ **(D)** $\frac{187}{192}$ **(E)** $\frac{63}{64}$

Problem 12.

For n a positive integer, let $f(n)$ be the quotient obtained when the sum of all positive divisors of n is divided by n. For example,

$$f(14) = (1+2+7+14) \div 14 = \frac{12}{7}.$$

What is $f(768) - f(384)$?

(A) $\frac{1}{768}$ **(B)** $\frac{1}{192}$ **(C)** 1 **(D)** $\frac{4}{3}$ **(E)** $\frac{8}{3}$

Problem 13.

Let $c = \frac{2\pi}{11}$. What is the value of

$$\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c}?$$

(A) -1 **(B)** $\frac{\sqrt{-11}}{5}$ **(C)** $\frac{\sqrt{11}}{5}$ **(D)** $\frac{10}{11}$ **(E)** 1

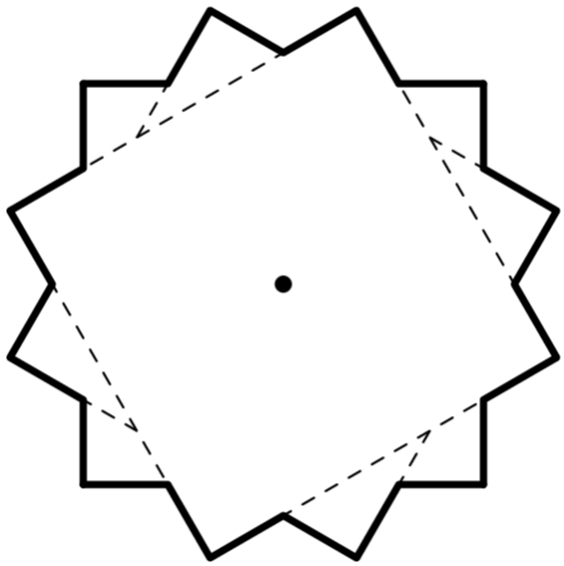
Problem 14.

Suppose that $P(z)$, $Q(z)$, and $R(z)$ are polynomials with real coefficients, having degrees 2, 3, and 6, respectively, and constant terms 1, 2, and 3, respectively. Let N be the number of distinct complex numbers z that satisfy the equation $P(z) \cdot Q(z) = R(z)$. What is the minimum possible value of N?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 5

Problem 15.

Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below. The area of this polygon can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?



(A) 75 **(B)** 93 **(C)** 96 **(D)** 129 **(E)** 147

Problem 16.

Let a, b, and c be positive integers such that $a + b + c = 23$ and

$$\gcd(a, b) + \gcd(b, c) + \gcd(c, a) = 9.$$

What is the sum of all possible distinct values of $a^2 + b^2 + c^2$?

(A) 259 **(B)** 438 **(C)** 516 **(D)** 625 **(E)** 687

Problem 17.

A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position?

(A) $\frac{13}{108}$ **(B)** $\frac{7}{54}$ **(C)** $\frac{29}{216}$ **(D)** $\frac{4}{27}$ **(E)** $\frac{1}{16}$

Problem 18.

Set $u_0 = \frac{1}{4}$, and for $k \geq 0$ let u_{k+1} be determined by the recurrence $u_{k+1} = 2u_k - 2u_k^2$. This sequence tends to a limit, call it L. What is the least value of k such that

$$|u_k - L| \leq \frac{1}{2^{1000}}?$$

(A) 10 **(B)** 97 **(C)** 253 **(D)** 329 **(E)** 401

Problem 19.

Regular polygons with $5, 6, 7$, and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

(A) 52 **(B)** 56 **(C)** 60 **(D)** 64 **(E)** 68

Problem 20.

A cube is constructed from 4 white unit cubes and 4 black unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Problem 21.

For real numbers x, let

$$P(x) = 1 + \cos(x) + i\sin(x) - \cos(2x) - i\sin(2x) + \cos(3x) + i\sin(3x)$$

where $i = \sqrt{-1}$. For how many values of x with $0 \le x < 2\pi$ does $P(x) = 0$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Problem 22.

Right triangle ABC has side lengths $BC = 6$, $AC = 8$, and $AB = 10$. A circle centered at O is tangent to line BC at B and passes through A. A circle centered at P is tangent to line AC at A and passes through B. What is OP?

(A) $\frac{23}{8}$ **(B)** $\frac{29}{10}$ **(C)** $\frac{35}{12}$ **(D)** $\frac{73}{25}$ **(E)** 3

Problem 23.

What is the average number of pairs of consecutive integers in a randomly selected subset of 5 distinct integers chosen from the set $\{1, 2, 3, \ldots, 30\}$? (For example the set $\{1, 17, 18, 19, 30\}$ has 2 pairs of consecutive integers.)

(A) $\frac{2}{3}$ **(B)** $\frac{29}{36}$ **(C)** $\frac{5}{6}$ **(D)** $\frac{29}{30}$ **(E)** 1

Problem 24.

Triangle ABC has side lengths $AB = 11, BC = 24$, and $CA = 20$. The bisector of $\angle BAC$

intersects $\overline{BC}$ in point D, and intersects the circumcircle of $\triangle ABC$ in point $E \neq A$. The circumcircle of $\triangle BED$ intersects the line AB in points B and $F \neq B$. What is CF?

(A) 28 **(B)** $20\sqrt{2}$ **(C)** 30 **(D)** 32 **(E)** $20\sqrt{3}$

Problem 25.

For n a positive integer, let $R(n)$ be the sum of the remainders when n is divided by 2, 3, 4, 5, 6, 7, 8, 9, and 10. For example, $R(15) = 1+0+3+0+3+1+7+6+5 = 26$. How many two-digit positive integers n satisfy $R(n) = R(n+1)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

1 AMC 12B 2021/1

What is the value of $1234 + 2341 + 3412 + 4123$?

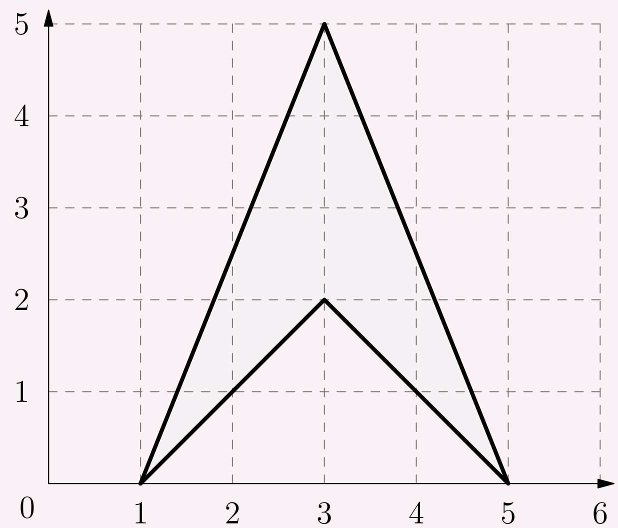
(A) $10,000$ **(B)** $10,010$ **(C)** $10,110$ **(D)** $11,000$ **(E)** $11,110$

Solution. $\boxed{\text{E}}$

ADD!

2 AMC 12B 2021/2

What is the area of the shaded figure shown below?



(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

Solution. $\boxed{\text{B}}$

COUNT!

3 AMC 12B 2021/3

At noon on a certain day, Minneapolis is N degrees warmer than St. Louis. At 4:00 the temperature in Minneapolis has fallen by 5 degrees while the temperature in St. Louis has risen by 3 degrees, at which time the temperatures in the two cities differ by 2 degrees. What is the product of all possible values of N?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 30 \qquad \textbf{(C)}\ 60 \qquad \textbf{(D)}\ 100 \qquad \textbf{(E)}\ 120$

Solution. $\boxed{\text{C}}$

At 4:00, the temperature in Minneapolis can be $\pm 2^\circ$ warmer than St. Louis. Therefore, at noon, Minneapolis is $(5+3) \pm 2$ degrees warmer than St. Louis, giving us $N = 10$ or $N = 6$.

4 AMC 12B 2021/4

Let $n = 8^{2022}$. Which of the following is equal to $\frac{n}{4}$?

(A) 4^{1010} **(B)** 2^{2022} **(C)** 8^{2018} **(D)** 4^{3031} **(E)** 4^{3032}

Solution. $\boxed{\text{E}}$

$n = 8^{2022} = 2^{6066} = 4^{3033} \implies n/4 = 4^{3032}$.

5 AMC 12B 2021/5

Call a fraction $\frac{a}{b}$, not necessarily in the simplest form special if a and b are positive integers whose sum is 15. How many distinct integers can be written as the sum of two, not necessarily different, special fractions?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

Solution. $\boxed{\text{C}}$

Ugly problem. bash

6 AMC 12B 2021/6

The largest prime factor of 16384 is 2, because $16384 = 2^{14}$. What is the sum of the digits of the largest prime factor of 16383?

(A) 3 **(B)** 7 **(C)** 10 **(D)** 16 **(E)** 22

Solution. $\boxed{\text{C}}$

$(2^{14} - 1) = (2^7 - 1)(2^7 + 1) = (127)(129) = 3 \cdot 43 \cdot 127.$

7 AMC 12B 2021/7

Which of the following conditions is sufficient to guarantee that integers x, y, and z satisfy the equation

$$x(x-y)+y(y-z)+z(z-x)=1?$$

(A) $x>y$ and $y=z$ **(B)** $x=y-1$ and $y=z-1$ **(C)** $x=z+1$ and $y=x+1$
(D) $x=z$ and $y-1=x$ **(E)** $x+y+z=1$

Solution. $\boxed{\text{D}}$

Plug in answer choices.

8 AMC 12B 2021/8

The product of the lengths of the two congruent sides of an obtuse isosceles triangle is equal to the product of the base and twice the triangle's height to the base. What is the measure, in degrees, of the vertex angle of this triangle?

(A) 105 **(B)** 120 **(C)** 135 **(D)** 150 **(E)** 165

Solution. $\boxed{\text{D}}$

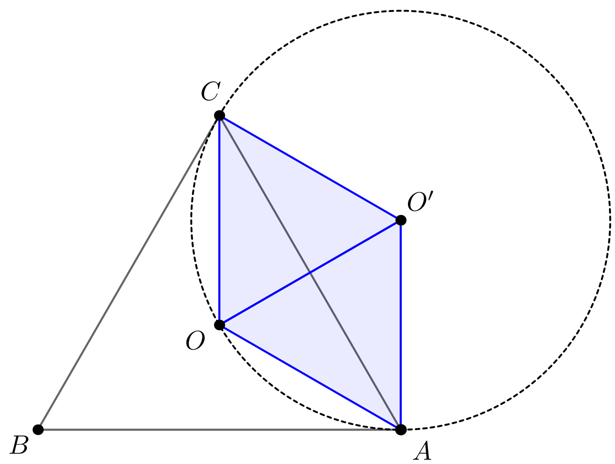
Let our triangle have side lengths x, x, b, height h, and vertex angle θ. The area of our triangle is equal to $\frac{1}{2}bh = \frac{1}{2}x^2 \sin\theta$. We're given that $x^2 = 2bh$, implying $\sin\theta = \frac{1}{2}$, so $\theta = 150^\circ$.

9 AMC 12B 2021/9

Triangle ABC is equilateral with side length 6. Suppose that O is the center of the inscribed circle of this triangle. What is the area of the circle passing through A, O, and C?

(A) 9π **(B)** 12π **(C)** 18π **(D)** 24π **(E)** 27π

Solution. $\boxed{\text{B}}$



Let O' be the reflection of O about AC. Note that $\triangle OO'C$ and $\triangle OO'A$ are both equilateral; therefore, O' is the center of (AOC). Our desired area is therefore equal to

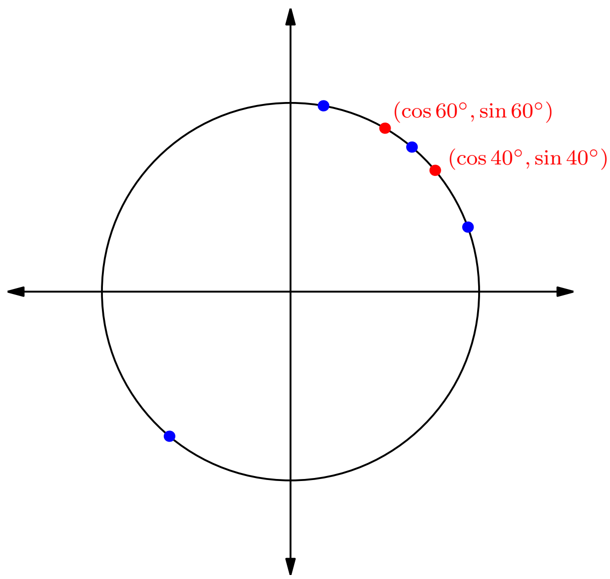
$$\pi\left(\frac{2}{3}\cdot\frac{\sqrt{3}}{2}\cdot 6\right)^2 = 12\pi.$$

10 AMC 12B 2021/10

What is the sum of all possible values of t between 0 and 360 such that the triangle in the coordinate plane whose vertices are $(\cos 40^\circ, \sin 40^\circ)$, $(\cos 60^\circ, \sin 60^\circ)$, and $(\cos t^\circ, \sin t^\circ)$ is isosceles?

(A) 100 **(B)** 150 **(C)** 330 **(D)** 360 **(E)** 380

Solution. $\boxed{\text{E}}$



Let X be the line segment connecting vertices $(\cos 40^\circ, \sin 40^\circ)$, and $(\cos 60^\circ, \sin 60^\circ)$. If X is a leg of our triangle, then $t = 80^\circ$ or $t = 20^\circ$; otherwise, X is the base, whence $t = 50^\circ$ or $t = 230^\circ$. Therefore, the total sum of all possibilities is 380.

11 AMC 12B 2021/11

Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

(A) $\frac{3}{4}$ **(B)** $\frac{57}{64}$ **(C)** $\frac{59}{64}$ **(D)** $\frac{187}{192}$ **(E)** $\frac{63}{64}$

Solution. $\boxed{\text{C}}$

If the product has no factors of 2, then all rolls must have been 1, 3, or 5, which occurs with probability $(1/2)^6$. If the product has one factor of 2, then exactly one roll must have been 2 or 6, while all others were 1, 3, or 5 as before. This occurs with probability $2 \cdot 6 \cdot (1/2)^5$. Therefore, our desired probability is equal to

$$1 - \left(\frac{1}{2}\right)^6 - 12\left(\frac{1}{2}\right)^5 = \frac{59}{64}.$$

12 AMC 12B 2021/12

For n a positive integer, let $f(n)$ be the quotient obtained when the sum of all positive divisors of n is divided by n. For example,

$$f(14) = (1 + 2 + 7 + 14) \div 14 = \frac{12}{7}.$$

What is $f(768) - f(384)$?

(A) $\frac{1}{768}$ **(B)** $\frac{1}{192}$ **(C)** 1 **(D)** $\frac{4}{3}$ **(E)** $\frac{8}{3}$

Solution. $\boxed{\text{B}}$

We have $768 = 2^8 \cdot 3$, and $384 = 2^7 \cdot 3$. Therefore,

$$\begin{aligned} f(768) - f(384) &= \frac{(1 + 2 + \ldots + 2^8)(1 + 3)}{2 \cdot 384} - \frac{(1 + 2 + \ldots + 2^7)(1 + 3)}{384} \\ &= \frac{(2 + 2^2 + \ldots + 2^9) - (2^2 + \ldots + 2^9)}{384} = \frac{1}{192}. \end{aligned}$$

13 AMC 12B 2021/13

Let $c = \frac{2\pi}{11}$. What is the value of

$$\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c}?$$

(A) -1 **(B)** $\frac{\sqrt{-11}}{5}$ **(C)** $\frac{\sqrt{11}}{5}$ **(D)** $\frac{10}{11}$ **(E)** 1

Solution. $\boxed{\text{E}}$

Observe that $\sin 3c = \sin 3c$, $\sin 6c = -\sin 5c$, $\sin 9c = -\sin 2c$, $\sin 12c = \sin c$, $\sin 15c = \sin 4c$, so our answer is E.

14 AMC 12B 2021/14

Suppose that $P(z)$, $Q(z)$, and $R(z)$ are polynomials with real coefficients, having degrees 2, 3, and 6, respectively, and constant terms 1, 2, and 3, respectively. Let N be the number of distinct complex numbers z that satisfy the equation $P(z) \cdot Q(z) = R(z)$. What is the minimum possible value of N?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 5

Solution. $\boxed{\text{B}}$

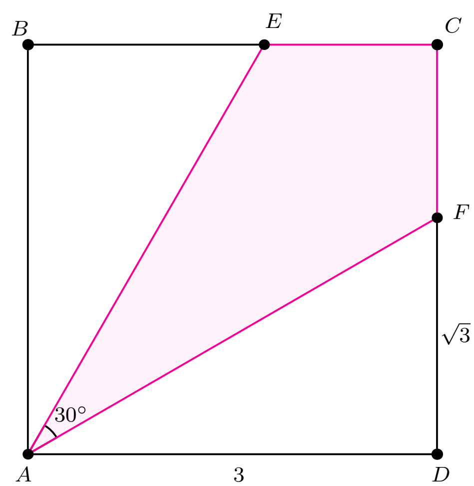
Because P, Q, R are polynomials with real coefficients, $N \geq 1$. We can show that this bound is tight by letting $R(x) = (x-1)^6 + (x^3+2)(x^2+1)$, $Q(x) = x^3 + 2$, $P(x) = x^2 + 1$, so that $R(x) - P(x) \cdot Q(x) = (x-1)^6$ only has one distinct root.

15 AMC 12B 2021/15

Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below. The area of this polygon can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?

(A) 75 **(B)** 93 **(C)** 96 **(D)** 129 **(E)** 147

Solution. $\boxed{\text{E}}$



Note that the polygon comprises twelve copies of the region shaded in pink depicted above. In the notation of the diagram, we know that $\angle EAF = 30^\circ$ (each square is rotated through this angle), and that AC bisects $\angle EAF$, so we know that $\angle FAD = \angle BAE = 30^\circ$. Therefore, our total area is equal to

$$12(3^2 - 3\sqrt{3}) = 108 - 36\sqrt{3}.$$

16 AMC 12B 2021/16

Let a, b, and c be positive integers such that $a + b + c = 23$ and

$$\gcd(a, b) + \gcd(b, c) + \gcd(c, a) = 9.$$

What is the sum of all possible distinct values of $a^2 + b^2 + c^2$?

(A) 259 **(B)** 438 **(C)** 516 **(D)** 625 **(E)** 687

Solution. $\boxed{\text{B}}$

Parity analysis gives us that a, b, c must all be odd. If $(\gcd(a, b), \gcd(b, c), \gcd(c, a)) = (7, 1, 1)$, then we have $(a, b, c) = (7, 7, 9)$ as a valid solution. If $(\gcd(a, b), \gcd(b, c), \gcd(c, a)) = (5, 3, 1)$, then we have $(a, b, c) = (5, 15, 3)$ as another solution. It is easy to see that no other possibilities work, so our answer is

$$(7^2 + 7^2 + 9^2) + (5^2 + 15^2 + 3^2) = 438.$$

17 AMC 12B 2021/17

A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position?

(A) $\frac{13}{108}$ **(B)** $\frac{7}{54}$ **(C)** $\frac{29}{216}$ **(D)** $\frac{4}{27}$ **(E)** $\frac{1}{16}$

Solution. $\boxed{\text{A}}$

For any valid path, our bug is confined to seven points: the starting point, and the ring of six points directly surrounding the starting point. Let "U" denote any step from the center to the outer ring, "D" denote any step from the outer ring to the center, and "S" denote any step along the outer ring. Note that for any given step, if the bug is along the outer ring, then there is one possible way to make a D move and two possible ways to make an S move; if the bug is in the middle, then it has six possible ways to make a U move.

Case 1: One U step. In this case, because the first step must be a U, then the bugs path can read $USSSD$ or $USSSS$, for a total of $6 \cdot 2^3 + 6 \cdot 2^4$ paths.

Case 2: Two U steps. In this case, a DU must be part of the bugs path. If there is one D, then we have three ways to place the DU (e.g $UDUSS$), giving us $3 \cdot 6^2 \cdot 2^2$ paths, and if we have two Ds, then we have two ways to place the DU (the very last step must be a D), giving us an additional $2 \cdot 6^2 \cdot 2$ paths.

Case 3: Three U steps. The only possible sequence is $UDUDU$, giving us 6^3 paths.

Finally, our total probability is

$$\frac{6 \cdot 24 + 6^2 \cdot 16 + 6^3}{6^5} = \frac{13}{108}.$$

18 AMC 12B 2021/18

Set $u_0 = \frac{1}{4}$, and for $k \geq 0$ let u_{k+1} be determined by the recurrence $u_{k+1} = 2u_k - 2u_k^2$. This sequence tends to a limit, call it L. What is the least value of k such that

$$|u_k - L| \leq \frac{1}{2^{1000}}?$$

(A) 10 **(B)** 97 **(C)** 253 **(D)** 329 **(E)** 401

Solution. $\boxed{\text{A}}$

We can use induction to prove that $u_k = \frac{2^{2^k}-1}{2^{2^k+1}}$ (note that this gives us $L = \frac{1}{2}$). The base case $k = 0$ holds. Assume that our equation holds for some $k = \ell$, and we wish to show that our equation holds for $k + 1 = \ell + 1$. Let $m = 2^\ell$ for brevity; then,

$$\begin{aligned}
u_{\ell+1} &= 2\left(\frac{2^m - 1}{2^{m+1}}\right) - 2\left(\frac{2^m - 1}{2^{m+1}}\right)^2 \\
&= \frac{(2^{m+1} - 2)(2^{m+1}) - 2(2^m - 1)^2}{2^{2m+2}} \\
&= \frac{2^{2m} - 1}{2^{2m+1}} \\
&= \frac{2^{2^{\ell+1}} - 1}{2^{2^{\ell+1}+1}},
\end{aligned}$$

which proves our inductive conclusion.

Finally, we want to find $|u_k - \frac{1}{2}| \leq \frac{1}{2^{1000}}$, which simplifies to $\frac{1}{2^{2^k}+1} \leq \frac{1}{2^{1000}}$, so $k \geq 10$.

19 AMC 12B 2021/19

Regular polygons with $5, 6, 7,$ and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

(A) 52 **(B)** 56 **(C)** 60 **(D)** 64 **(E)** 68

Solution. $\boxed{\text{E}}$

Consider the intersections between the pentagon and the hexagon. Each time a side of the hexagon "crosses" a vertex of the pentagon, two intersections are added. Because the pentagon has less sides than the hexagon, the number of times this can happen is limited by the number of sides of the pentagon, which is 5; therefore, there are 10 intersections (we are given that we don't have to worry about triple intersections). Similarly, for all $\{m, n\}$, the number of intersections between the two polygons with m, n sides, respectively, when inscribed in the same circle is $2\min(m, n)$. Summing over all unordered pairs, this gives us a final answer of

$$3 \cdot 10 + 2 \cdot 12 + 1 \cdot 14 = 68.$$

20 AMC 12B 2021/20

A cube is constructed from 4 white unit cubes and 4 black unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Solution. $\boxed{\text{A}}$

Careful casework: all four white cubes grouped together (1), three white cubes in an "L" shape (4), two bars of two white cubes each (1), and all white cubes separated in a checkerboard pattern (1).

21 AMC 12B 2021/21

For real numbers x, let

$$P(x) = 1 + \cos(x) + i\sin(x) - \cos(2x) - i\sin(2x) + \cos(3x) + i\sin(3x)$$

where $i = \sqrt{-1}$. For how many values of x with $0 \le x < 2\pi$ does $P(x) = 0$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution. $\boxed{\text{A}}$

Note that $P(x) = 1 + \omega - \omega^2 + \omega^3$, where $\omega = e^{ix}$. Because this is a polynomial with real coefficients, $P(x) = (\omega - r)(\omega^2 \pm ... \pm 1)$ for some r, justified by the fact that P has complex conjugate roots (both with magnitude 1, since $|\omega| = 1$). By comparing coefficients, this forces $|r| = 1$; however, because $P(0)$, $P(\pi) \neq 0$, this means that there are no solutions in x to the given polynomial.

Alternatively, this problem can be solved by setting

$$\sin x + \sin 3x = \sin 2x,$$

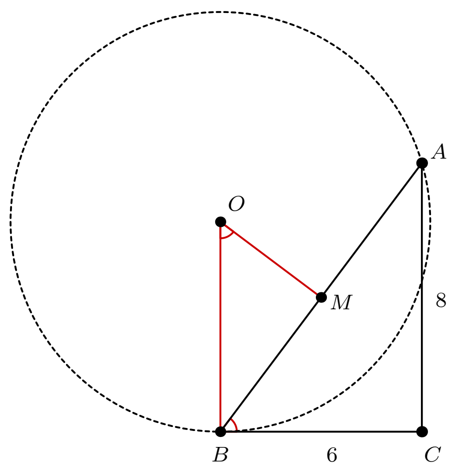
which must hold in order for the complex part of the polynomial to vanish. Using the sum-to-product formula, this is equivalent to $\cos x = 1/2$ or $\sin x = 0$. None of the four possibilities for x satisfy $P(x) = 0$; therefore, there are no solutions.

22 AMC 12B 2021/22

Right triangle ABC has side lengths $BC = 6$, $AC = 8$, and $AB = 10$. A circle centered at O is tangent to line BC at B and passes through A. A circle centered at P is tangent to line AC at A and passes through B. What is OP?

(A) $\frac{23}{8}$ **(B)** $\frac{29}{10}$ **(C)** $\frac{35}{12}$ **(D)** $\frac{73}{25}$ **(E)** 3

Solution. $\boxed{\text{C}}$



Let M be the midpoint of AB, and ω be the circle centered at O passing through A,B. Note that O is the intersection between the perpendicular line to BC at B, and the perpendicular bisector of AB. Also, we have

$$\angle ABC = \frac{1}{2}\angle BOA = \angle BOM,$$

since ω is tangent to BC. It follows that $\triangle BOM \sim \triangle ABC$, so that

$$OM = \frac{BM \cdot BC}{AC} = \frac{15}{4}.$$

Using a similar line of reasoning, we calculate $PM = \frac{20}{3}$, giving us $OP = \frac{35}{12}$.

23 AMC 12B 2021/23

What is the average number of pairs of consecutive integers in a randomly selected subset of 5 distinct integers chosen from the set $\{1, 2, 3, \ldots, 30\}$? (For example the set $\{1, 17, 18, 19, 30\}$ has 2 pairs of consecutive integers.)

(A) $\frac{2}{3}$ **(B)** $\frac{29}{36}$ **(C)** $\frac{5}{6}$ **(D)** $\frac{29}{30}$ **(E)** 1

Solution. $\boxed{\text{A}}$

There are $\binom{5}{2}$ different pairs of numbers in each set. For each of these pairings, they have probability $29/\binom{30}{2} = \frac{1}{15}$ of being consecutive integers; therefore, by the linearity of expectation, the expected number of pairs being consecutive integers in each set is equal to

$$10 \cdot \frac{1}{15} = \frac{2}{3}.$$

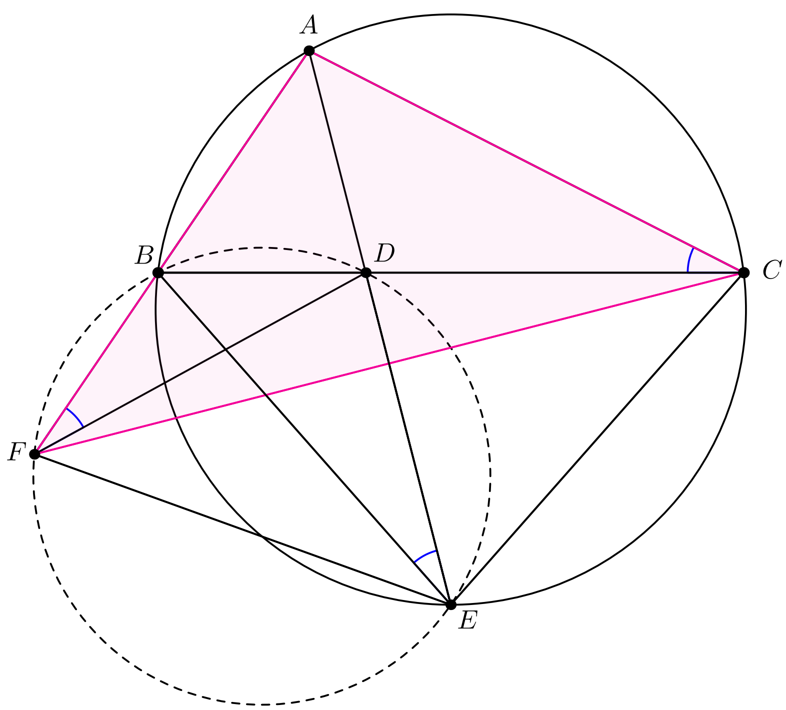
24 AMC 12B 2021/24

Problem 24.

Triangle ABC has side lengths $AB = 11$, $BC = 24$, and $CA = 20$. The bisector of $\angle BAC$ intersects $\overline{BC}$ in point D, and intersects the circumcircle of $\triangle ABC$ in point $E \neq A$. The circumcircle of $\triangle BED$ intersects the line AB in points B and $F \neq B$. What is CF?

(A) 28 **(B)** $20\sqrt{2}$ **(C)** 30 **(D)** 32 **(E)** $20\sqrt{3}$

Solution. $\boxed{\text{C}}$



By virtue of $BDEF$ and $ABCD$ being cyclic quadrilaterals, we have that

$$\angle BFD = \angle DEB = \angle AEB = \angle ACB = \angle ACD.$$

Therefore, $\triangle AFD \cong \triangle ACD$, giving us $AF = AC$ and $\triangle FAC$ isosceles. Now, we can compute:

$$CF = 2 \cdot AC \sin\left(\frac{\angle FAC}{2}\right) = 2 \cdot AC\sqrt{\frac{1 - \left(\frac{BC^2 - AC^2 - AB^2}{-2 \cdot AC \cdot AB}\right)}{2}} = 2 \cdot 20 \cdot \left(\frac{3}{4}\right) = 30.$$

25 AMC 12B 2021/25

For n a positive integer, let $R(n)$ be the sum of the remainders when n is divided by 2, 3, 4, 5, 6, 7, 8, 9, and 10. For example, $R(15) = 1+0+3+0+3+1+7+6+5 = 26$. How many two-digit positive integers n satisfy $R(n) = R(n+1)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution. $\boxed{\text{C}}$

Suppose none of the integers $2 \leq k \leq 10$ divides $n+1$. Then, we would have that

$$d_n = R(n+1) - R(n) = 9,$$

since each of the 9 remainders would increase by 1.

For each j that actually ends up dividing $n+1$, our value of d_n decreases by j, since the remainder going from n to $n+1$ upon division by j decreases by $j-1$ (the remainder for n is $j-1$, while the remainder for $n+1$ is 0), rather than increasing by 1 as expected. Thus, we need to choose all valid combinations of integers from $2 \leq k \leq 10$ which divide $n+1$ such that their sum S_n is 9, which would force $d_n = 0$. Call an integer *good* if it divides $n+1$, and *bad* otherwise.

If 10 is good, $S_n \geq 10 > 9$, which violates our condition that $S_n = 9$. Therefore, we require that 10 is bad. If 9 is good, then 3 is necessarily also good, giving us $S_n \geq 12 > 9$; therefore, 9 is also bad. A similar line of reasoning can be used to show that 8 and 6 must be bad. If 7 is good, then we can have 2 be good, which gives us two possible solutions $n+1 = 7 \cdot 2$ and $n+1 = 7^2 \cdot 2$. Now, assuming that all $6-10$ are bad, it is easy to verify that no other combinations of good numbers from $1-5$ work, so we have $n = 13$ and $n = 97$ as our only two solutions.