PoTW 18: Week of 9-25-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #18: AMC Week 1

combo/geo?
Source: 2012 AMC 12A

Let S be the square one of whose diagonals has endpoints $(0.1, 0.7)$ and $(-0.1, -0.7)$. A point $v = (x, y)$ is chosen uniformly at random over all pairs of real numbers x and y such that $0 \leq x \leq 2012$ and $0 \leq y \leq 2012$. Let $T(v)$ be a translated copy of S centered at v. What is the probability that the square region determined by $T(v)$ contains exactly two points with integer coordinates in its interior?

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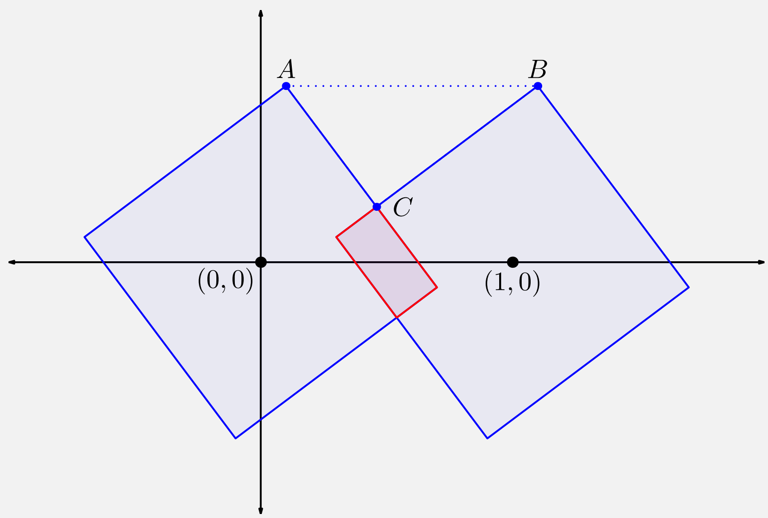
Solution (intended)**:**

Note that our square has side length $\frac{\sqrt{2}}{2}(\sqrt{0.2^2 + 1.4^2}) = 1$. Therefore, it suffices to consider $v \in \{0 \le x \le 1, 0 \le y \le 1\}$ (call this unit-square region $\mathcal{H}$).

Consider the region of overlap $\mathcal{R}$ between $T((0,0))$ and $T((1,0))$. For all points v_i in $\mathcal{R}$, $T(v_0)$ contains both $(0,0)$ and $(1,0)$; moreover, for all v_i not in $\mathcal{R}$, then $T(v_0)$ cannot contain both points. This follows from the fact that, for any two identical squares X and Y in the coordinate plane, the center of X is contained in the region bounded by Y if and only if the center of Y is contained in the region bounded by X.



Thus, in the diagram above, we desire the area of the region of the red rectangle above the x-axis. Because $AB = 1$, we have that $AC = 0.6$ and $BC = 0.8$ (for example, by considering $\angle ABC$), from which it follows that the dimensions of our rectangle are 0.2 and 0.4. Because there are four of these half-rectangles in $\mathcal{H}$, our final answer is

$$4\left(\frac{1}{2} \cdot 0.2 \cdot 0.4\right) = \frac{4}{25}.$$

Here's a nice video explanation by Richard Rusczyk: https://youtu.be/fm36K88S0LU.