PoTW 20: Week of 11-19-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #20: Poly Fib

Algebra
Source: Princeton HS

Define the usual Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2},$$

for all $n > 1$, $F_0 = 0$ and $F_1 = 1$. If we fix $P(0) = k$, then there exists a unique real polynomial $P(x)$ with degree at most 9 such that $P(2n) = F_{2n+1}$ for $1 \leq n \leq 9$. What is the value of k that minimizes the expression $P(0)^2 + P(20)^2$?

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First, a brief exposition on finite differences: given any polynomial $f(x)$ with degree d, consider the sequence $f(a), f(a+1), f(a+2), \ldots$, for some arbitrary a. Then, we define the following sequences of finite differences:

- 1st sequence: $f(a+1)-f(a)$, $f(a+2)-f(a+1)$, $f(a+3)-f(a+2)$, ...
- 2nd sequence: $f(a+2)-2f(a+1)+f(a)$, $f(a+3)-2f(a+2)+f(a+1)$, ...
- ...

We create each new sequence by taking the difference between adjacent terms in the previous sequence.

For concreteness, consider the example $f(x) = x^2$ and $a = 0$. Then, our original sequence reads $0, 1, 4, 9, 16, 25, 36, \ldots$, giving us the following sequences of finite differences:

- 1st sequence: $1, 3, 5, 7, 9, 11, \ldots$
- 2nd sequence: $2, 2, 2, 2, 2, \ldots$
- 3rd sequence: $0, 0, 0, 0, 0, \ldots$

Important properties of these sequences:

Theorem 1 (Finite Differences). Let $f(x)$ be a polynomial with degree d and leading coefficient a_d.

(a). The i-th term of the n-th sequence of finite differences is given by

$$\sum_{k=0}^{n}(-1)^n\binom{n}{k}f(a+k+i)$$

(b). All elements of the d-th sequence are equal to the constant c, where $c = a_d \cdot d!$. All elements of the $d+1$-th sequence are 0.

Both parts of this theorem can be proven inductively. For part (a), adjacent terms of the sequence will have coefficients for each f(something) that sum easily with pascal's identity. For part (b), if we suppose that $f(x) = a_d x^d + \ldots$, then

$$f(x+1)-f(x) = a_d(x+1)^d - a_d(x)^d + \ldots = a_d(d)x^{d-1} + \ldots,$$

etc. and we can continue inducting downwards.

We're now ready to present our solution. Both use finite differences; the first utilizes part (b) and a manual computation of each sequence of finite differences, while the second uses a direct application of the formula in part (a).

Solution 1 (benq):

Consider the sequence $k, F_3, F_5, \dots, F_{19}$, corresponding to $P(0), P(2), \dots, P(18)$. Then:

- the 1st sequence of finite differences: $(F_3 - k)$, F_4, F_6, ..., F_{18}
- the 2nd sequence of finite differences: $(F_4 - F_3 + k)$, F_5, F_7, ..., F_{17}
- the 3rd sequence of finite differences: $(F_5 - F_4 + F_3 - k)$, F_6, F_8, ..., F_{16}
- $\vdots$
- the 8th sequence of finite differences: $(F_{10} - F_9 + ... - F_3 + k)$, F_{11}.

By Theorem 1, we know that the 9th sequence is constant; therefore, the next term in the 8th sequence is equal to $F_{11} + (F_{11} - F_{10} + F_9 - ... + F_3 - k)$. This allows us to calculate the next term in each subsequent sequence "higher-up" in the ladder, so that we get that

$$\begin{aligned} P(20) + k &= (F_{11} - F_{10} + F_9 - ... F_3) + (F_{11} + F_{12} + ... F_{19}) \\ &= (F_{10} + 1) + (F_{21} - F_{12}) \\ &= F_{21} - F_{11} + 1. \end{aligned}$$

Therefore, $k = \frac{F_{21} - F_{11} + 1}{2} = 5429$.

Solution 2 (benq):

By Theorem 1, we have that

$$0 = \sum_{k=0}^{10} (-1)^n \binom{n}{k} P(2k),$$

which rearranges to

$$k + P(20) = \sum_{k=1}^{9} (-1)^{k+1} \binom{n}{k} F_{2k+1}.$$

Now ... messy algebra, details of which are left out here (:
Using Binet's formula and the binomial theorem, the right hand side collapses to $F_{21} - F_{11} + 1$, so done.