PoTW 21: Week of 11-26-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #21: Contrived Turkey

Algebra-ish
Source: inspired by Ankan

To prepare Thanksgiving for his girlfriend, Jay buys infinitely many turkeys. Suppose that each turkey can be represented by a uniform cube of side length 1. Jay labels the turkeys from 2 to ∞, and uses the following scheme to cut the turkey labeled i:

- line up the turkey so that from birds-eye perspective it fills up the square on the cartesian plane with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.
- make two continuously curvy cuts, one along the line $y = x$, and one along the line $y = \sqrt[i]{1-x^i}$, where i is the label of the turkey he is cutting.
- his cutting splits the turkey into four regions. he gives himself two of these regions; the region with center of mass closest to the x-axis, and the region with center of mass farthest away from the x-axis.

Assume that he follows this cutting process for all $i \geq 2$, and let A_i be the volume of the turkey that he gives to himself for each i. Compute

$$\sum_{k=2}^{\infty} \frac{(-A_k)^k}{k}.$$

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Solution (intended):

The function $f(x) = \sqrt[i]{1 - x^i}$ is symmetrical about the line $y = x$ for all integer $i \geq 2$. Also, the function $f(x) = x$ is symmetrical about the same line. Therefore, $A_k = 1/2$ for all $k \geq 2$, and our desired sum can be computed:

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{(-A_k)^k}{k} &= \sum_{k=2}^{\infty} \frac{(-1/2)^k}{k} \\ &= -\sum_{k=1}^{\infty} \frac{(-1/2)^{k+1}}{k} + \frac{1}{2} \\ &= \frac{1}{2} - \ln\left(\frac{3}{2}\right).\end{aligned}$$