PoTW 22: Week of 12-3-2021 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #22: Cool Ellipse problem

Geometry
Source: Tovi Wen

Let $\mathcal{E}$ be an ellipse with a major axis of length 10. Circle ω with center O is tangent to $\mathcal{E}$ at P. The line OP intersects the major and minor axes of $\mathcal{E}$ at X and Y, respectively. Suppose that $PX = 4$ and $PY = 6$. Find the distance between the foci of $\mathcal{E}$.

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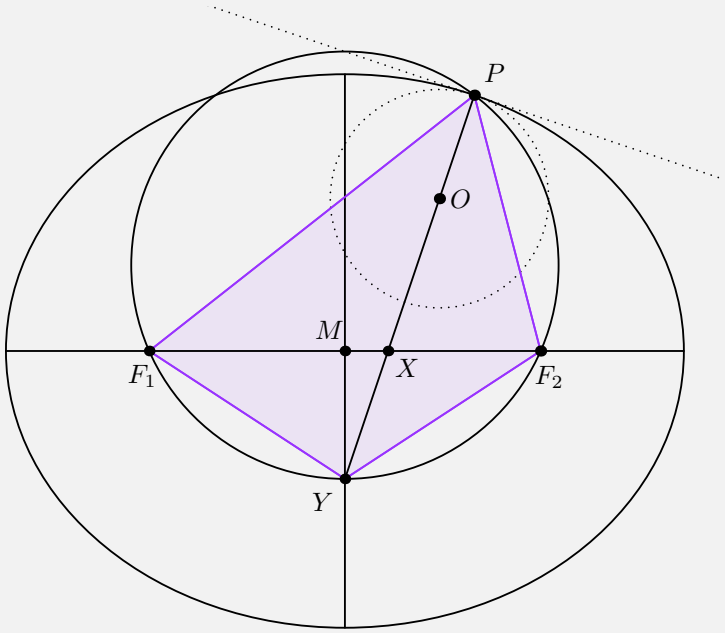
Solution:
*Solution equivalent to the submissions by **Jeffrey Chen**!*

The answer is $\boxed{10\sqrt{3}/3}$.



Let F_1 and F_2 be the two foci of $\mathcal{E}$. By the reflection property of ellipses, OP bisects $\angle F_1PF_2$. Because Y lies on the perpendicular bisector of F_1F_2, it follows that YF_1PF_2 has circumcircle ω.

Let M be the center of $\mathcal{E}$, $c = F_1M = F_2M$, and $d = MX$. By power of a point wrt X in ω, $(c+d)(c-d) = 8 \implies c^2 - d^2 = 8$. Moreover, by Ptolemy's, we have:

$$PF_1 \cdot YF_2 + PF_2 \cdot YF_1 = F_1F_2 \cdot PY = 12c.$$

But because $YF_1 = YF_2$, and $PF_1 + PF_2 = 10$, the above expression simplifies:

$$\begin{aligned} 12c &= 10 \cdot YF_1 \\ &= 10\sqrt{F_1M^2 + MY^2} \\ &= 10\sqrt{c^2 + 2^2 - d^2} \\ &= 20\sqrt{3}. \end{aligned}$$

Therefore, $2c = 10\sqrt{3}/3$, as desired.