PoTW 26: Week of 1-14-2022 (solution)*

Problem of the Week at shsmathteam.com

Problem of the Week #26: positive integers only

Algebra

Source: AIME

Consider the recursion:

$$a_{n+2} = \frac{a_n + 2022}{1 + a_{n+1}}$$

for the sequence $\{a_n\}_{n\geq 1}$. Given that all a_n are positive integers, compute the minimum possible value of $a_1 + a_2$.

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Solution (equivalent to **gf4848** on AoPS)**:**

We know that

$$a_{n+2} + a_{n+2}a_{n+1} = a_n + 2022.$$

By shifting indices, we also have that

$$a_{n+3} + a_{n+3}a_{n+2} = a_{n+1} + 2022.$$

Subtracting these two equations gives us that

$$a_{n+3} - a_{n+1} = \frac{a_{n+2} - a_n}{a_{n+2} + 1}.$$

Because each a_i are all positive integers, $b_i = a_{i+2} - a_i$ is also a positive integer sequence. On the other hand, the relationship above implies that b_i is strictly decreasing, which is impossible given that the first few terms of a_i are finite. Thus, we must have $b_i = 0$, so the problem reduces to minimizing $a_1 + a_2$ over all $a_1 a_2 = 2022$.

A word on motivation: motivating this solution comes more naturally by testing smaller cases. In any of the possibilities that you try for a_1, a_2, you'll find quickly that $a_3 = a_1$ is forced.